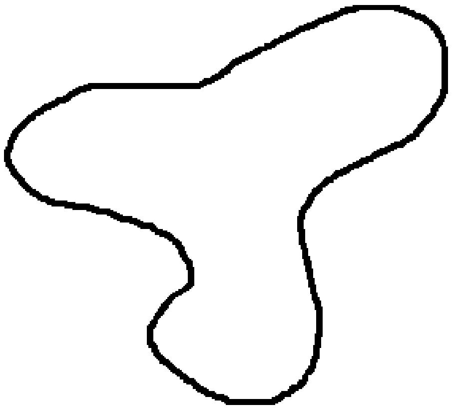
When you see this symbol

Copy the notes and diagrams into your jotter.

The area under a curve

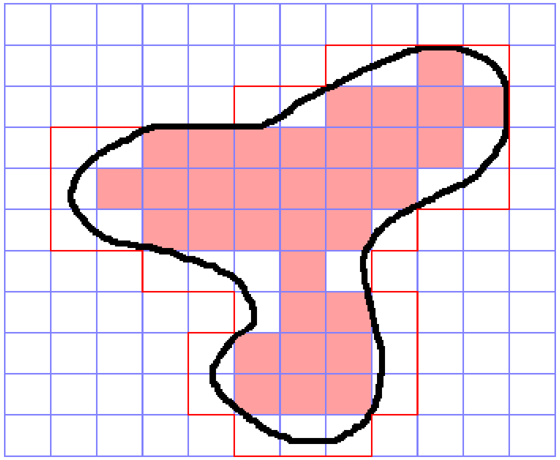
Let us first consider the irregular shape shown opposite.



How can we find the area A of this shape?

The area under a curve

We can find an approximation by placing a grid of squares over it.



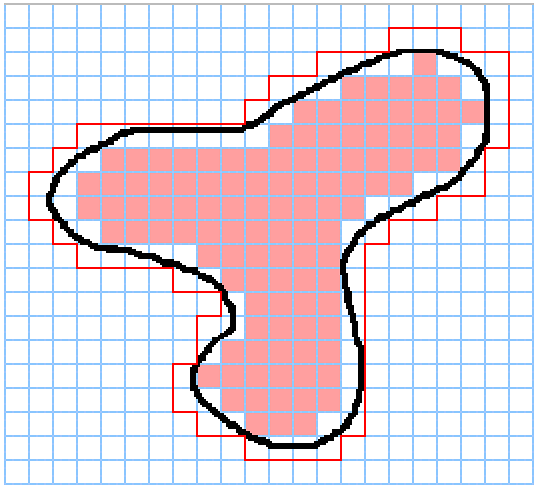
By counting squares,

$A > 33$ and $A < 60$

i.e. $33 < A < 60$

The area under a curve

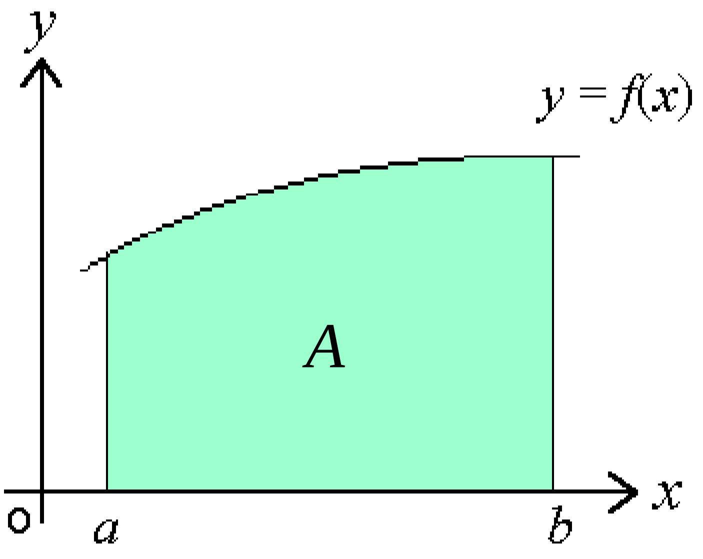
By taking a finer ‘mesh’ of squares we could obtain a better approximation for *A*.



We now study another way of approximating to *A*, using rectangles, in which *A* can be found by a limit process.

The area under a curve

The diagram shows part of the curve $y = f(x)$ from $x = a$ to $x = b$.

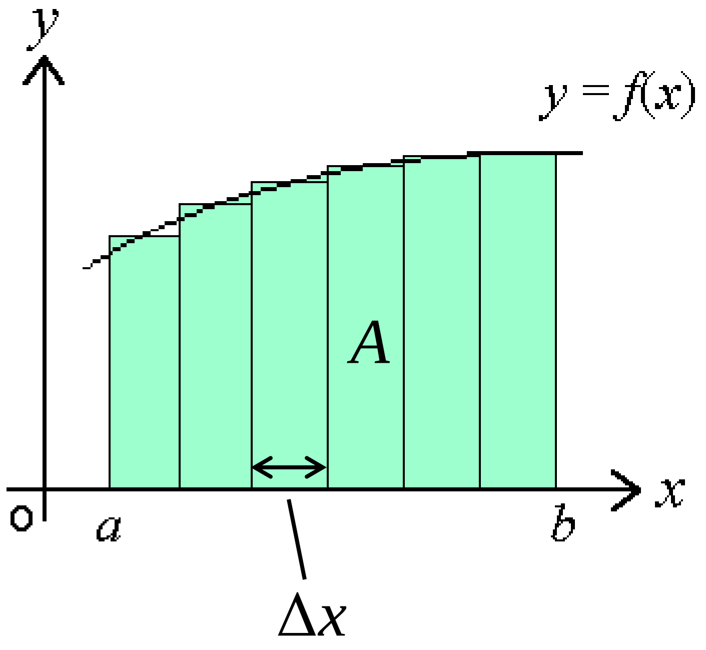


We will find an expression for the area A bounded by the curve, the x-axis, and the lines $x = a$ and $x = b$.

The area under a curve

The interval $[a,b]$ is divided into n sections of equal width, Δx.

n rectangles are then drawn to approximate the area A under the curve.

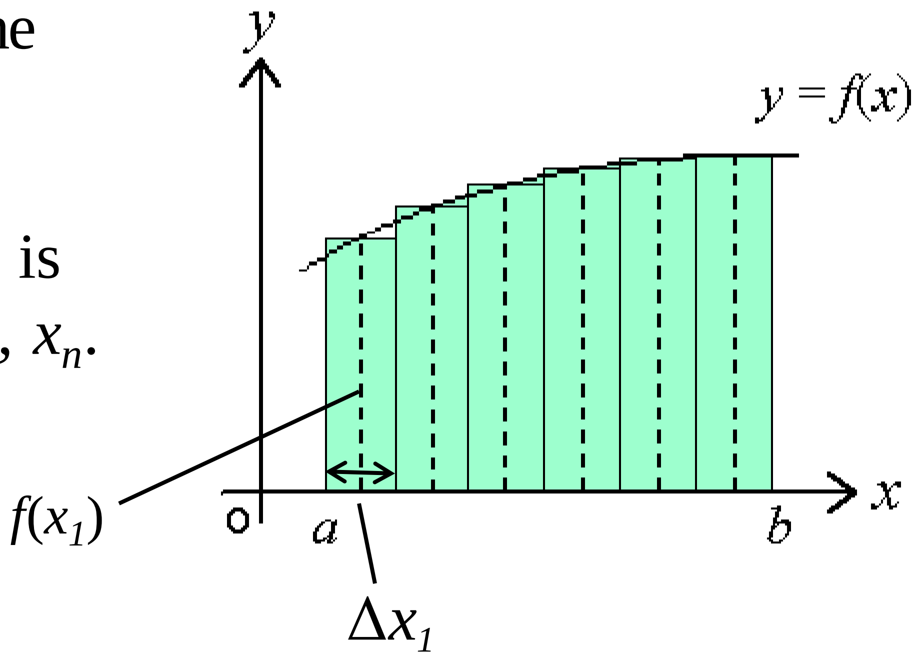


The area under a curve

Dashed lines represent the height of each rectangle.

The position of each line is given by an x-coordinate, x_n.

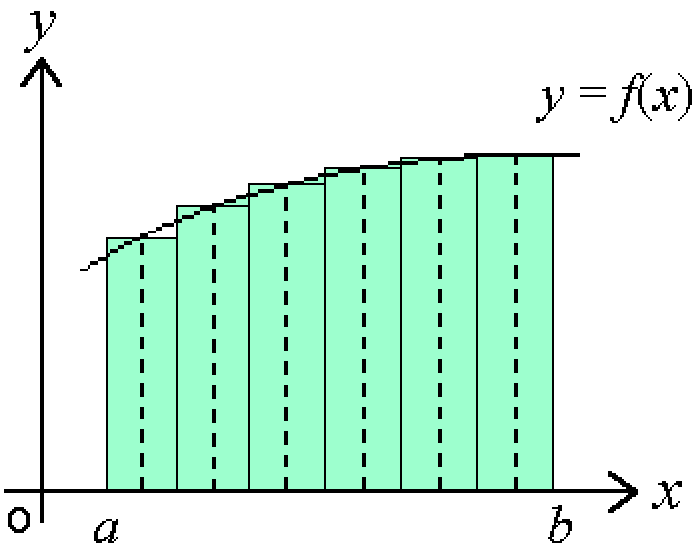
The first rectangle has height $f(x_1)$ and breadth Δx_1.



Thus the area of the first rectangle $= f(x_1).\Delta x_1$

The area under a curve

An approximation for the area under the curve, between $x = a$ to $x = b$, can be found by summing the areas of the rectangles.



$$A = f(x_1).\Delta x_1 + f(x_2).\Delta x_2 + f(x_3).\Delta x_3 + f(x_4).\Delta x_4 + f(x_5).\Delta x_5 + f(x_6).\Delta x_6$$

The area under a curve

Using the Greek letter Σ (sigma) to denote 'the sum of', we have

$$A \approx \sum_{i=1}^{i=6} f(x_i).\Delta x_i$$

For any number *n* rectangles, we then have

$$A \approx \sum_{i=1}^{i=n} f(x_i).\Delta x_i$$

The area under a curve

In order to emphasise that the sum extends over the interval [*a*,*b*], we often write the sum as

$$A \approx \sum_{x=b}^{x=a} f(x).\Delta x$$

The area under a curve

By increasing the number *n* rectangles, we decrease their breadth Δx.

As Δx gets increasingly smaller we say it ‘tends to zero’,

i.e. $\Delta x \to 0$.

So we define $$A = \lim_{\Delta x \to 0} \sum_{x=b}^{x=a} f(x).\Delta \mathrm{x}$$

The area under a curve

The form $$A = \lim_{\Delta x \to 0} \sum_{x=b}^{x=a} f(x).\Delta \mathrm{x}$$

was simplified into the form that we are familiar with today

$$A = \int_a^b f(x)dx$$

This reads

‘the area *A* is equal to the integral of *f*(*x*) from *a* to *b*’.

The area under a curve

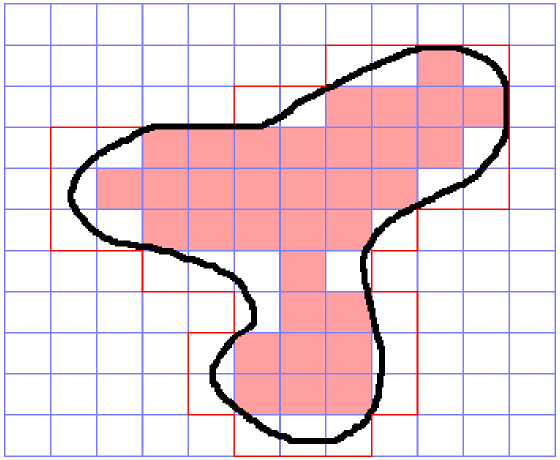
We have derived a method for finding the area under a curve and a formal notation

$$A = \int_a^b f(x)dx$$

We have seen the integration symbol $\int$ before in connection with anti-differentiation, but we have not yet connected finding the area under a curve with the process of integration.

The area under a curve

Let us remind ourselves of where we started.



Can we apply this method to calculate the area under a curve?

The area under a curve

Consider a strip under the curve *h* wide.

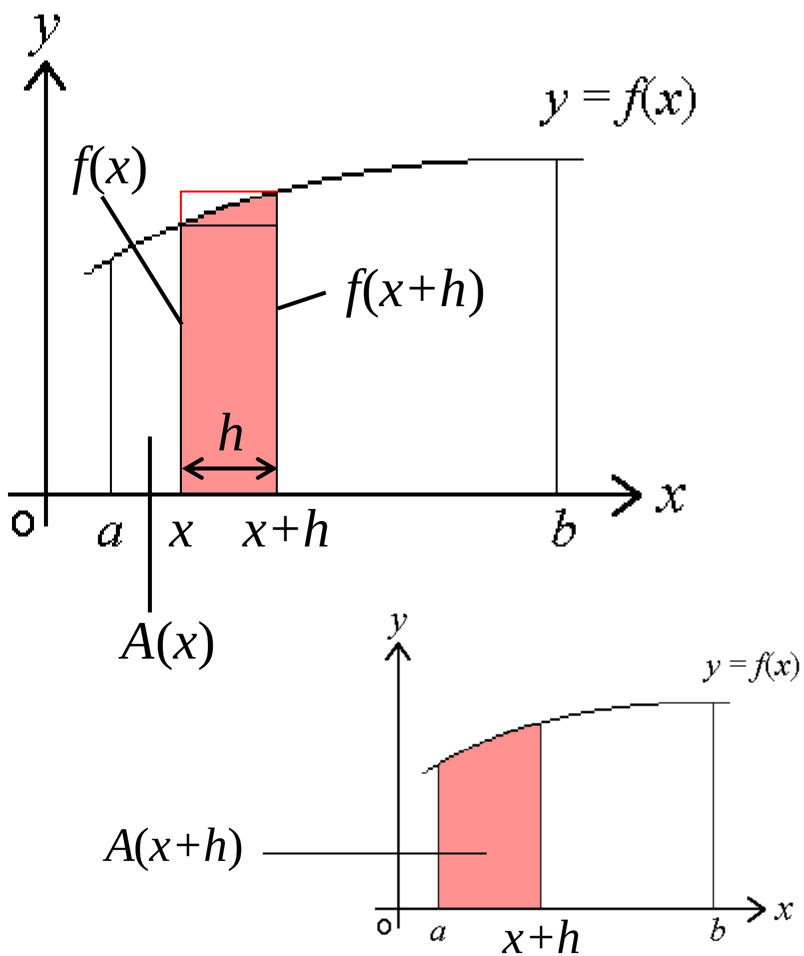
The inner rectangle has area

$h \times f(x)$.

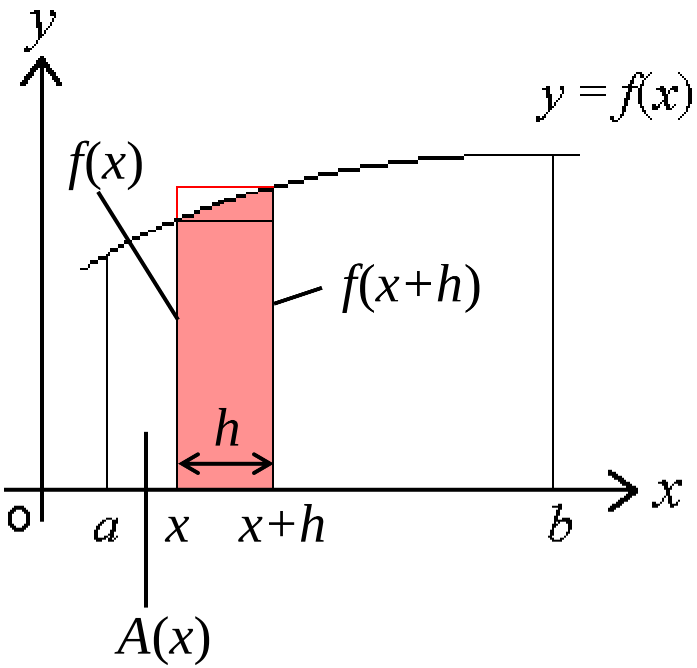
The outer rectangle has area

$h \times f(x+h)$.

The actual area is given by $A(x+h) - A(x)$.



The area under a curve



Comparing areas,

$$h \times f(x) \leq A(x+h) - A(x) \leq h \times f(x+h)$$

The area under a curve

$$h \times f(x) \leq A(x+h) - A(x) \leq h \times f(x+h)$$

Dividing by h ($\neq 0$), $$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h)$$

As h $\to$ 0, $$f(x) \leq \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \leq f(x)$$

$$\Rightarrow \qquad f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

So $f(x) = A'(x)$, by the definition of a derived function

and $A(x) = \int f(x) dx$, by the definition of integration.

The area under a curve

In conclusion,

the area A bounded by the x-axis, the lines $x = a$ and $x = b$ and the curve $y = f(x)$ is denoted by,

$$\int_a^b f(x)dx$$