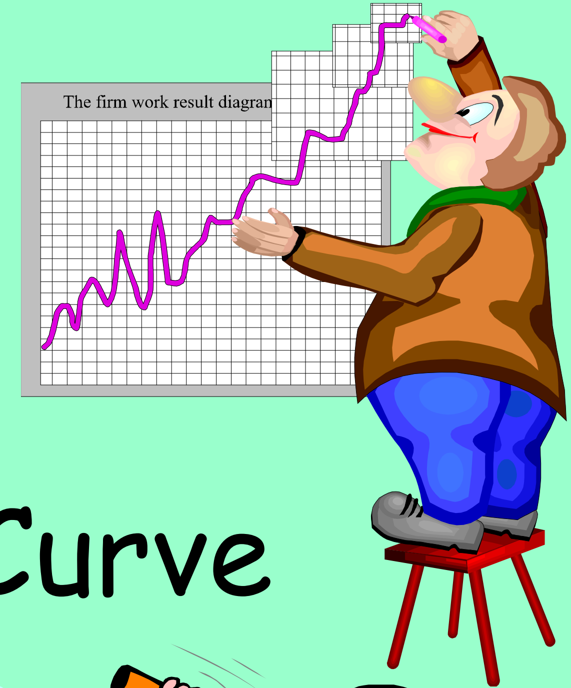
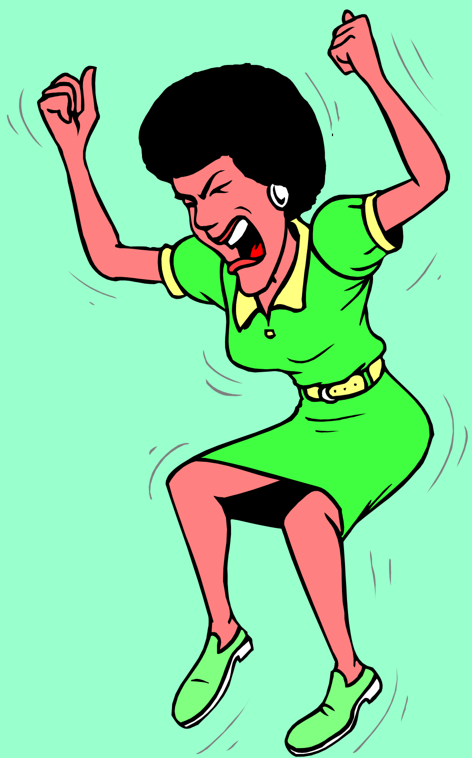
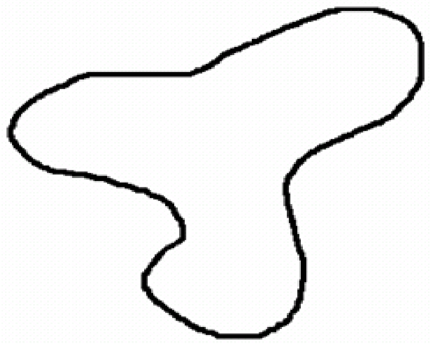


4. Area Under the Curve



The area under a curve

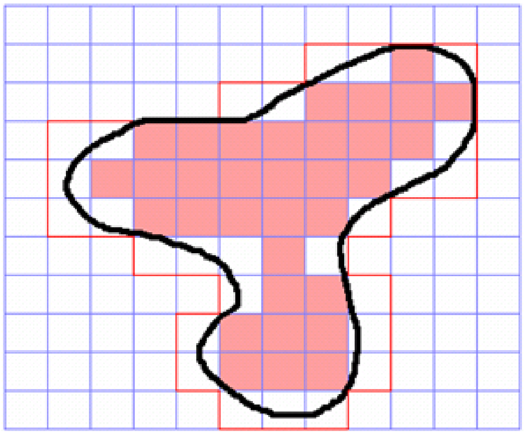
Let us first consider the irregular shape shown opposite.



How can we find the area A of this shape?

The area under a curve

We can find an approximation by placing a grid of squares over it.



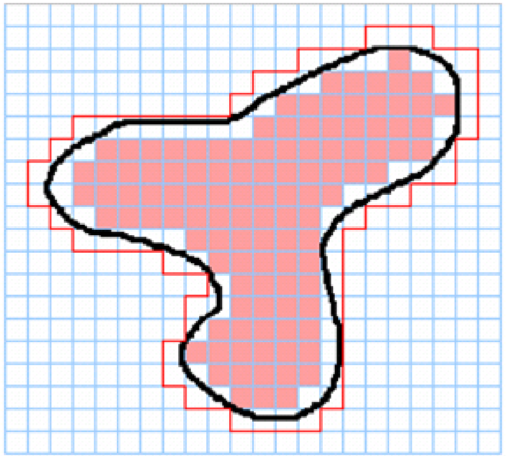
By counting squares,

$A > 33$ and $A < 60$

i.e. $33 < A < 60$

The area under a curve

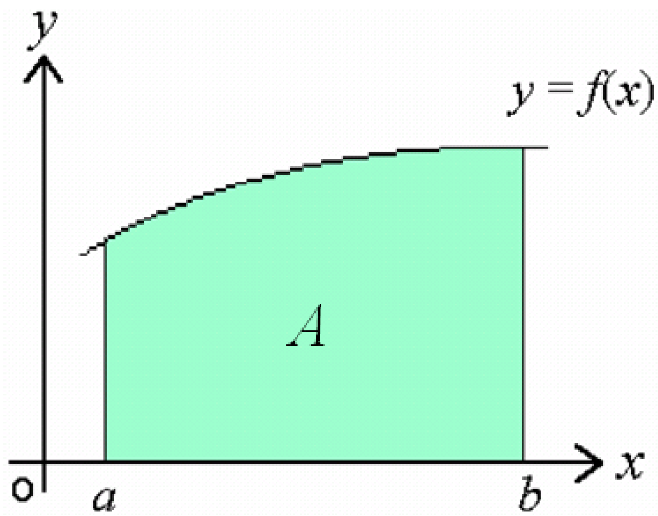
By taking a finer 'mesh' of squares we could obtain a better approximation for A.



We now study another way of approximating to A, using rectangles, in which A can be found by a limit process.

The area under a curve

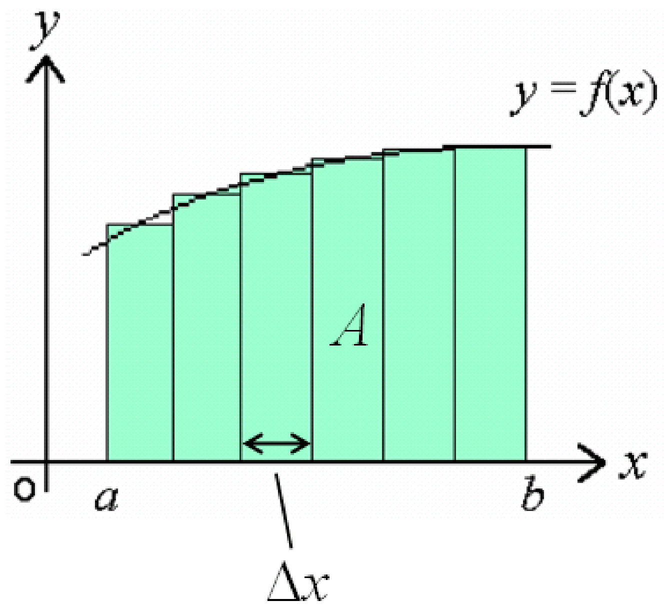
The diagram shows part of the curve $y = f(x)$ from $x = a$ to $x = b$.



We will find an expression for the area A bounded by the curve, the x-axis, and the lines $x = a$ and $x = b$.

The area under a curve

The interval [a,b] is divided into n sections of equal width, Δx.



n rectangles are then drawn to approximate the area A under the curve.

Look at me

The area under a curve

By increasing the number n rectangles, we decrease their breadth Δx.

As Δx gets increasingly smaller we say it 'tends to zero',

i.e. $\Delta x \to 0$.

So we define $$A = \lim_{\Delta x \to 0} \sum_{x=b}^{x=a} f(x).\Delta x$$

The area under a curve

The form $$A = \lim_{\Delta x \to 0} \sum_{x=b}^{x=a} f(x).\Delta \text{x}$$

was simplified into the form that we are familiar with today

$$A = \int_a^b f(x)dx$$

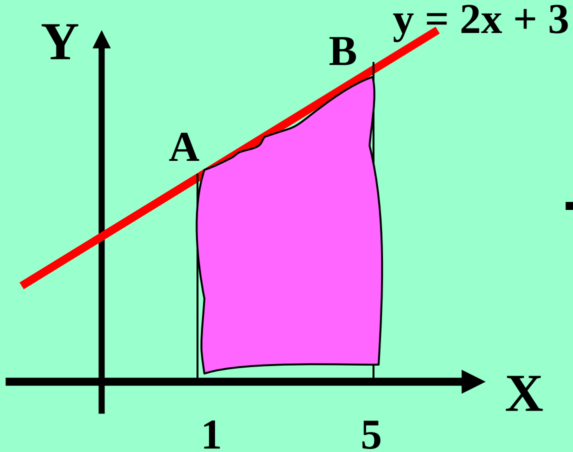
This reads

'the area A is equal to the integral of $f(x)$ from a to b'.

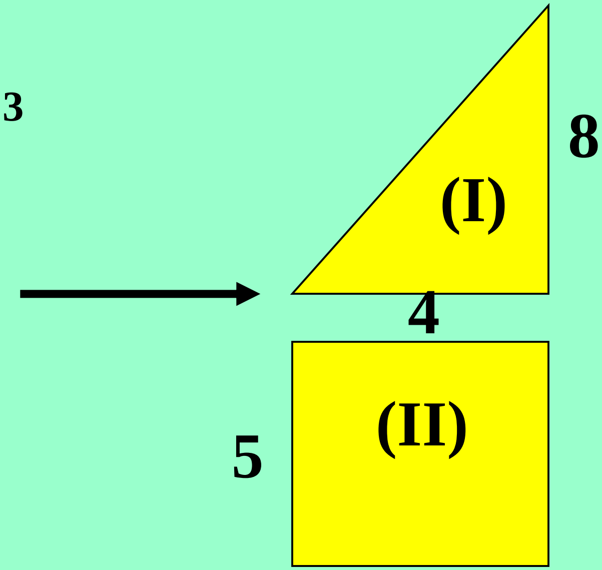
INTEGRATION & AREA

NB: **we should think of a line as a very simple curve!**

Consider



A is (1,5) & B is (5,13)



Area(I) = 8 X 4 ÷ 2= 16

Area(II) = 5 X 4 = 20

Total = 36

Now consider

$$\int_{1}^{5}(2x+3)dx = \left[x^2 + 3x\right]_1^5 = (25 + 15) - (1 + 3) = \underline{36}$$

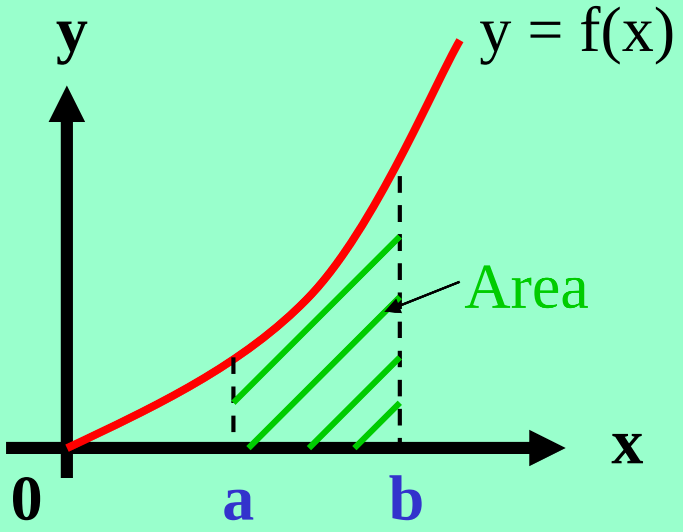
Comparing answers we should see that the area can also be obtained by integration so we can use this for curves as follows….

Finding the area between the graph and the x-axis

Copy the following:

The area between the graph of y= f(x) and the horizontal axis between x = a and x = b is:

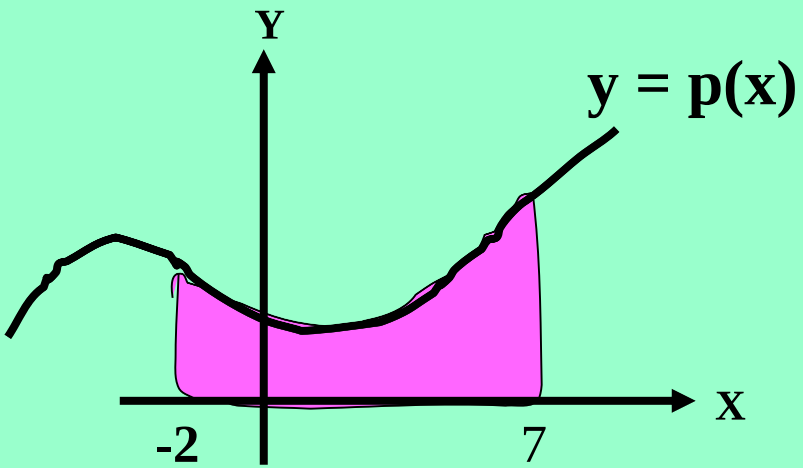
$$Area = \int_{a}^{b} f(x)dx$$



This is a definite integral with a as the lower limit and b as the upper limit.

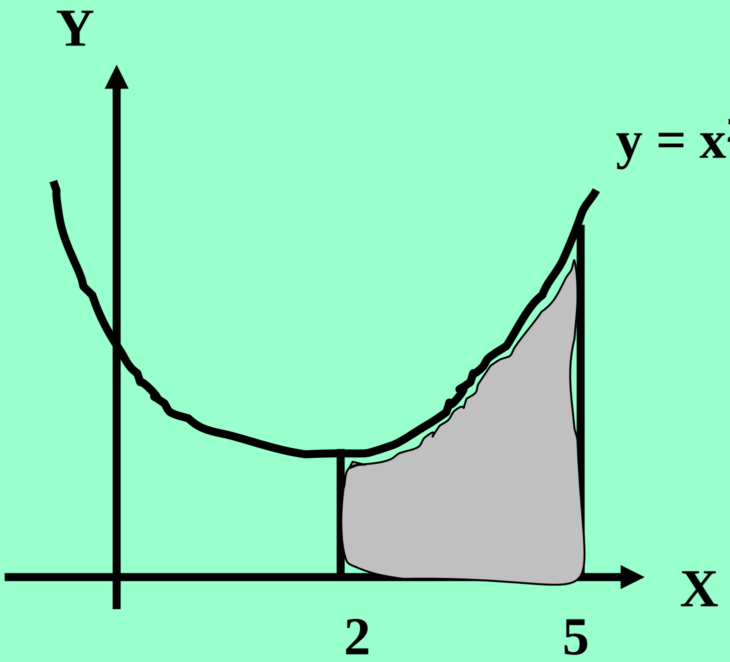
Area Under a Curve

Example 1



Shaded area = $\int_{-2}^{7} p(x)dx$

Example 2



Area = $\int_{2}^{5} (x^2 - 4x + 5)\,dx$

$$= \left[\frac{x^3}{3} - \frac{4x^2}{2} + 5x \right]_2^5$$

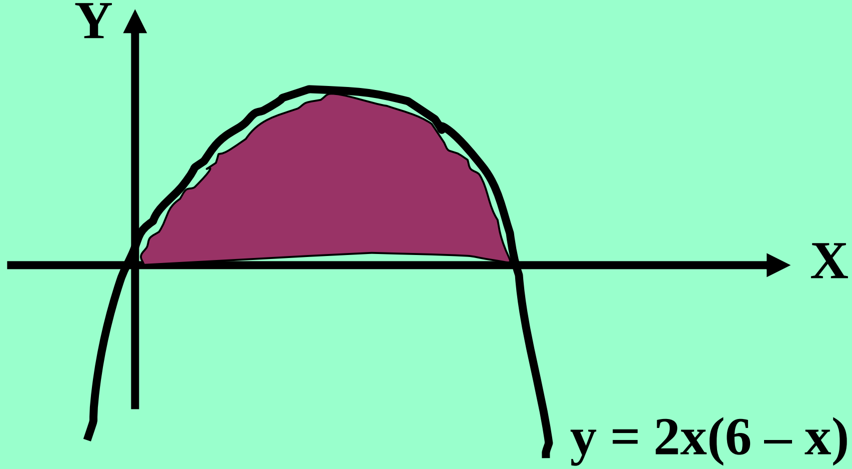
$$= \left[\tfrac{1}{3}x^3 - 2x^2 + 5x \right]_2^5$$

$$= \left(\tfrac{125}{3} - 50 + 25\right) - \left(\tfrac{8}{3} - 8 + 10\right)$$

= 12 units²

Example 3

NAB



NB: need limits!

1. Find limits by setting f(x) = 0 and factorising

Curve cuts X-axis when $2x(6 - x) = 0$ so $x = 0$ or $x = 6$

2. Prepare and then evaluate integral between these limits

$$Area = \int_0^6 (2x(6-x))\,dx$$

$$= \int_0^6 (12x - 2x^2)\,dx$$

$$= \left[\, 6x^2 - {}^{2}/_{3}x^3 \,\right]_0^6$$

$$= (216 - 144) - 0$$

$$= \underline{72 \text{ units}^2}$$

Heinemann, p.167, EX 9K, Q1