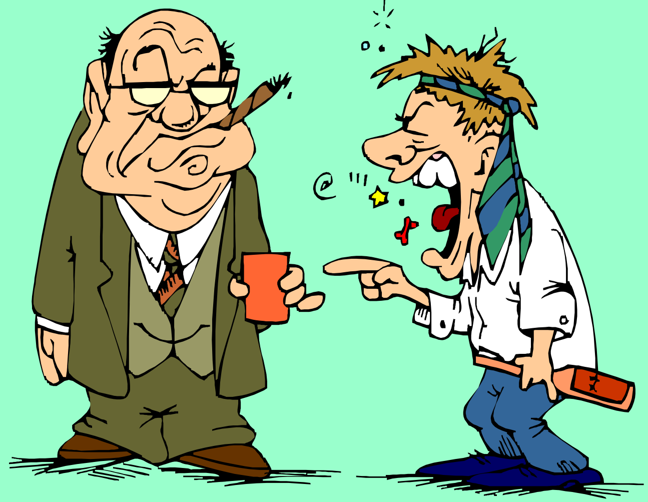
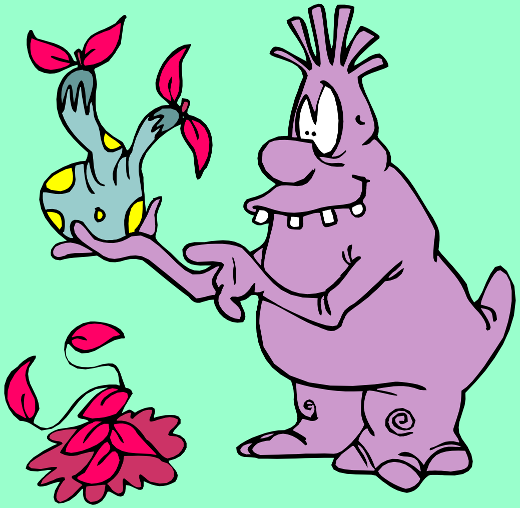




3. Linear Recurrence Relations

$$U_{n+1} = mU_n + c$$

Linear Recurrence Relations

Recurrence Relations can take more than one form. For example:

$u_{n+1} = 5u_n$ or $u_{n+1} = u_n - 6$

Linear recurrence relations have a particular form:

$$u_{n+1} = mu_n + c$$

Think: y = mx + c

Can you see why this is called a linear recurrence relation?

In most questions involving recurrence relations **you will have to identify three things:**

$$u_{n+1} = mu_n + c$$

1. Starting value, u_0

2. Multiplier

3. Addition or subtraction

Linear Recurrence Relations Problems

Example 1

Jim opens a savings account with £1000. At
year he pays in another £5000. The rate of int

Increase : multiplier bigger than 1

Decrease: multiplier less than 1

(a) How much is in the account after 4 years?

(b) How many years is it before he has £50 000 in the account?

Solution to part (a): **you will have to identify three things:**

1. Starting value, u_0 — $u_0 = 1000$

2. Multiplier — Increase is 4.5% pa → Multiplier = 1.045

3. Addition or subtraction — He **ADDS £5000** each year

Recurrence relation is: $u_{n+1} = 1.045u_n + 5000 \quad (u_0 = 1000)$

Linear Recurrence Relations Problems

Example 1

Jim opens a savings account with £1000. At the beginning of each year he pays in another £5000. The rate of interest is fixed at 4.5%.

(a) How much is in the account after 4 years?

(b) How many years is it before he has £50 000 in the account?

Solution to part (a):

$$u_{n+1} = 1.045u_n + 5000 \qquad (u_0 = 1000)$$

$$u_0 = 1000$$

u_1 = 1.045 x 1000 + 5000 = £6045

u_2 = 1.045 x 6045 + 5000 = £11317.025

u_3 = 1.045 x u_2 + 5000 = £16826.29113

u_4 = 1.045 x u_3 + 5000 = £22,583.47423

Having found U_1 use "ANS" Key then X1.045 + 5000 on calculator

Balance after 4 years = £22,583.47

Linear Recurrence Relations Problems

Example 1

Jim opens a savings account with £1000. At the beginning of each year he pays in another £5000. The rate of interest is fixed at 4.5%.

(a) How much is in the account after 4 years?

(b) How many years is it before he has £50 000 in the account?

Solution to part (b):

$u_4 = 1.045 \times u_3 + 5000 \quad = £22{,}583.47423$

$u_5 = 1.045 \times u_4 + 5000 \quad = £28{,}599.73057$

$u_6 = 1.045 \times u_5 + 5000 \quad = £34{,}886.71844$

$u_7 = 1.045 \times u_6 + 5000 \quad = £41{,}456.62077$

$u_8 = 1.045 \times u_7 + 5000 \quad = £48{,}322.16871$

$u_9 = 1.045 \times u_8 + 5000 \quad = £55{,}496.6663$

It takes 9 years to accumulate £50,000

Continue sequence whenever questions ask “when will value be X?”

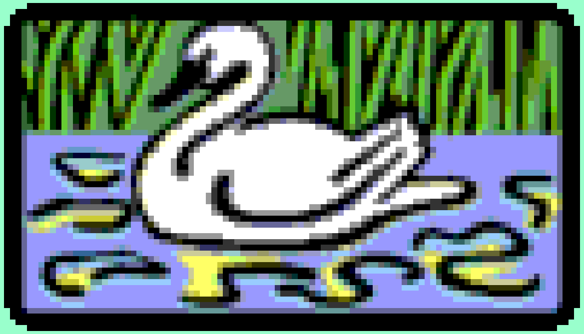
Example 2

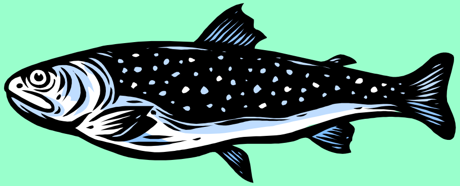
A factory wishes to dump 150kg of a particular waste product into a local stream once per week.

The flow of the water removes 60% of this material from the stream bed each week.

However it has been calculated that if the level of deposit on the stream bed reaches 265kg then there will be a serious risk to the aquatic life.

Should the factory be allowed to dump this waste indefinitely?





Example 2

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However it has been calculated that if the level of deposit on the stream bed reaches 265kg then there will be a serious risk to the aquatic life.

Should the factory be allowed to dump this waste indefinitely?

Solution :

1. Starting value, u_0 $\quad u_0 = 0$

We always want what is LEFT

2. Multiplier $\quad$ **40%** is Left each week → m = 0.4

3. Addition or subtraction $\quad$ Will **ADD 150 kg** per week

Recurrence relation is: $u_{n+1} = 0.4u_n + 150$

Example 2

Solution : Recurrence relation is: $u_{n+1} = 0.4u_n + 150$ $u_0 = 0$

$u_0 = 0$

$u_1 = 0.4 \times 0 + 150$ $= 150$

$u_2 =$ $= 210$

$u_3 =$ $= 234$

$u_4 =$ $= 243.6$

$u_5 =$ $= 247.44$

$u_6 =$ $= 248.976$

$u_7 =$ $= 249.5904$

$u_8 =$ $= 249.83616$

$u_9 =$ $= 249.934464$

$u_{10} =$ $= 249.9737856$

Example 2

Solution :

Recurrence relation is: $u_{n+1} = 0.4u_n + 150$

At this stage we should see that the values are staying between 249 and 250.

U			
11	249.9895142	21	249.9999989
12	249.9958057	22	249.9999996
13	249.9983223	23	249.9999998
14	249.9993289	24	249.9999999
15	249.9997316	25	250
16	249.9998926	26	250
17	249.9999571	27	250
18	249.9999828	28	250
19	249.9999931	29	250
20	249.9999973	30	250

Example 2

Solution : Recurrence relation is: $u_{n+1} = 0.4u_n + 150$

When amount of waste reaches 250kg it stays at this.

Check: If $u_n = 250$ then $u_{n+1} = 0.4 \times 250 + 150 = 250$

This is below the danger level of 265 Kg so factory could be allowed to continue dumping.

NB: We say that the sequence CONVERGES ("tends") to a LIMIT of 250 as $n \to \infty$

Example 3

A credit card company charges 2% per month on any outstanding debt, while a minimum fixed sum of £30 has to be repaid at the end of each month.

If £200 is owed initially, and no other purchases are made, how many months will it take before the debt is paid off.

Solution :

1. Starting value, u_0 — $u_0 = 200$
2. Multiplier — 2% added monthly → $m = 1.02$
3. Addition or subtraction — Debt **reduced** by £30 per month

Recurrence relation is: $u_{n+1} = 1.02u_n - 30$

Example 3

Solution :

Recurrence relation is: $u_{n+1} = 1.02u_n - 30$ $u_0 = 200$

$u_0 = 200$

$u_1 = 1.02 \times 200 - 30$ = 174

$u_2 =$ = 147.48

$u_3 =$ = 120.4296

$u_4 =$ = 92.838192

$u_5 =$ = 64.69495584

$u_6 =$ = 35.98885496

$u_7 =$ = 6.708632056

$u_8 =$ = -23.1571953

So debt will be repaid after 8 months

Booklet , p.6, EX 5