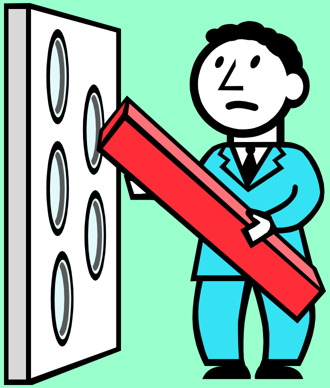
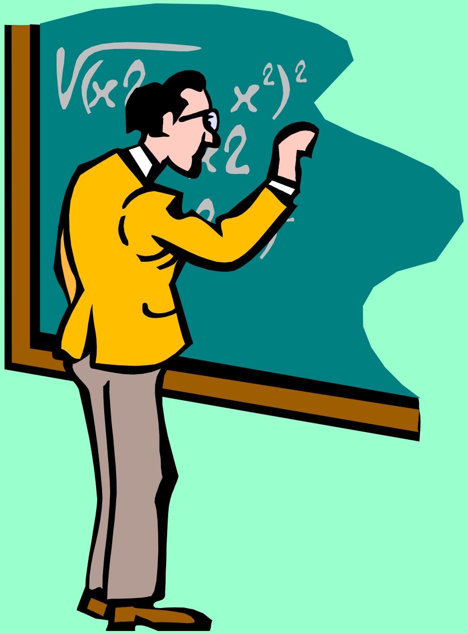
3. Completing the square

$$y = k(x - a)^2 + b$$





The forms of quadratics

Quadratics can be represented by the following forms:

$$y = ax^2 + bx + c$$

$$y = k(x-a)(x-b)$$ ← a and b are roots

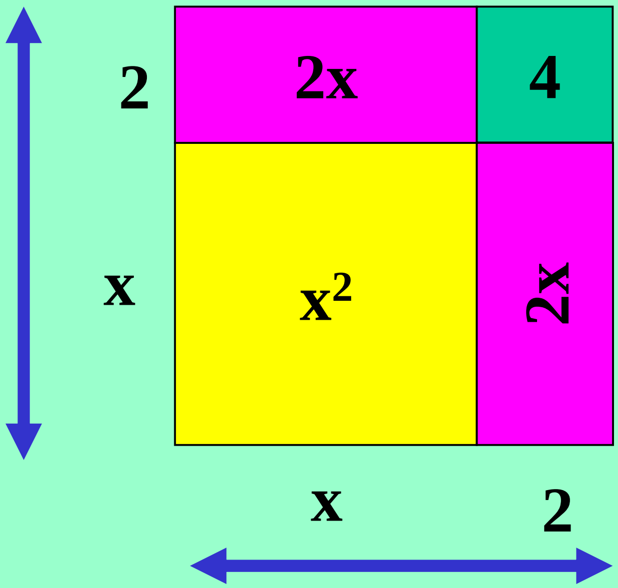
$$y = a(x+p)^2 + q$$ ← "completed square" form

The first two are not new to us so we are going to look at the third form.

Why "completed square"?

Consider the following function: $x^2 + 4x - 3$

If we consider this as being made up of areas of squares:



So $x^2 + 4x = (x + 2)\ (x + 2) - 4$

So $x^2 + 4x - 3 = [(x + 2)^2 - 4] - 3$

$= (x + 2)^2 - 7$

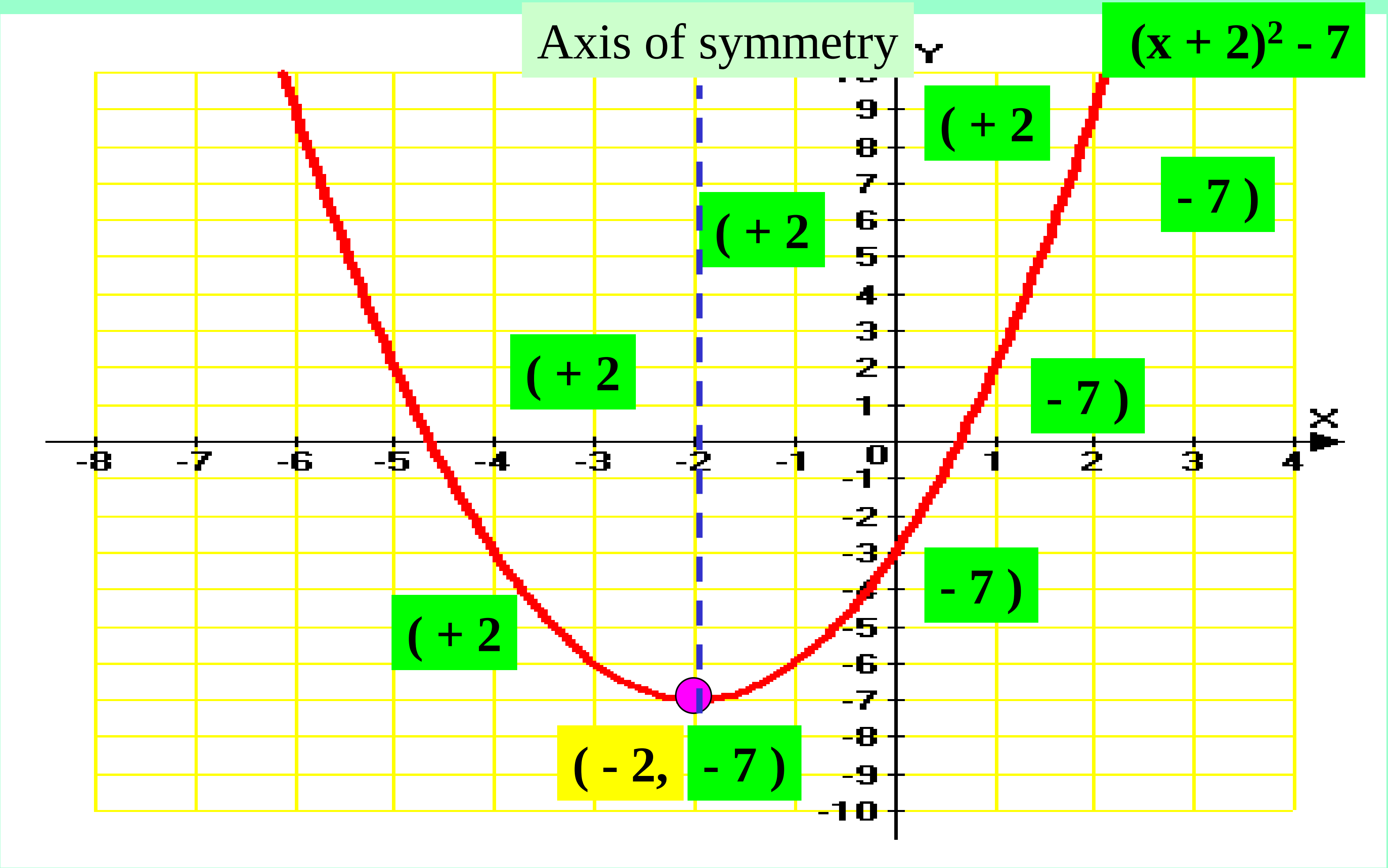
So far we have $x^2 + 4x$ but we don't have a square.

Why use the "completed square" form?

The completed square form is useful if we need to find the equation of a parabola which does not cut the x-axis (and so we don't have roots).

This is because we can read essential details from the graph or from the completed square form.

Take the graph of the equation we have just found: $\mathbf{(x + 2)^2 - 7}$



When in completed square form, we can determine the tp of the parabola from the equation.

The axis of symmetry = x-coord of tp

Example 1 (ALTERNATIVE)

Find the coordinates of the turning point for the curve with equation

$$y = x^2 - 4x - 12$$

Solution:

1. Expand general form

$$y = a(x + p)^2 + q$$

$$= a(x^2 + 2px + p^2) + q$$

$$= ax^2 + 2apx + ap^2 + q$$

2. Compare coefficients in expanded form with those for a, b and c in original.

$$y = x^2 - 4x - 12$$

a = 1

2ap = -4

2p = -4

p = -2

ap^2 + q = -12

$(-2)^2$ + q = -12

q = -16

3. TP occurs at (-p , q). So write down coordinates remembering +ve x^2 means happy parabola.

Change sign

So minimum tp at: (2 , -16)

Heinemann, p.146, EX 8D,
Q4, (a), (b), (c) & (d)

Example 2

Find the coordinates of the turning point and hence the axis of symmetry for the function: $14-3(2x-5)^2$

Solution:

1. Rewrite equation with bracket first (Optional).

2. Completed square form requires single x so divide bracket by any coeffiecient of x.

3. Read off coordinates of tp remembering to change sign of x coordinate.

4. Axis of symmetry occurs at x coordinate of tp.

$$=-3(2x-5)^2+14$$

$$a(x+p)^2+q$$

$$=-\tfrac{3}{2}(x-\tfrac{5}{2})^2+14$$

So: Max tp at $\left(\tfrac{5}{2},14\right)$

Axis of symmetry is :

$$x=\tfrac{5}{2}$$

Heinemann, p.146, EX 8D, Q1

Example 3 (ALTERNATIVE)

Find the coordinates of the turning point for the curve with equation

$$y = 3x^2 - 6x + 4$$

Solution:

1. Expand general form

2. Compare coefficients in expanded form with those for a, b and c in original.

3. TP occurs at (-p , q).
So write down coordinates
Remembering +ve x^2 means happy parabola.

$$y = a(x+p)^2 + q$$

$$= a(x^2 + 2px + p^2) + q$$

$$= ax^2 + 2apx + ap^2 + q$$

$$y = 3x^2 - 6x + 4$$

a = 3	2ap = -6	ap^2 + q = 4
	6p = -6	$3(-1)^2$ + q = 4
	p = -1	q = 1

Change sign

So minimum tp at: (1 , 1)

Heinemann, p.146, EX 8D, Q4, (e) & (f)

Example 4 (ALTERNATIVE)

By expressing $x^2 - 6x + 12$ in the form $(x+a)^2 + b$ find the maximum value of $\dfrac{12}{x^2 - 6x + 12}$

Solution:

1. Expand general form

2. Compare coefficients in expanded form with those for a, b and c in original.

3. Re- write fraction and use fact that all squares +ve

$$y = a(x+p)^2 + q$$

$$= ax^2 + 2apx + ap^2 + q$$

$$x^2 - 6x + 12$$

a = 1

2ap = -6

2p = -6

p = -3

$ap^2 + q = 4$

$(-3)^2 + q = 12$

q = 3

$(x - 3)^2 + 3$

Example 4 (ALTERNATIVE)

By expressing $x^2 - 6x + 12$ in the form $(x+a)^2 + b$ find the maximum value of $\dfrac{12}{x^2 - 6x + 12}$

$(x - 3)^2 + 3$

Solution:

So: $\dfrac{12}{x^2 - 6x + 12}$

$= \dfrac{12}{(x-3)^2 + 3}$

For x^2 lowest possible value is zero

$\text{max} = \dfrac{12}{3}$

3. Re- write fraction and use fact that all squares +ve

maximum = 4

Heinemann, p.146, EX 8D,

Q6