12.

Optimisation : Maxima & Minima

Problem Solving

Differentiation can be used to solve problems which require maximum or minimum values.

Problems typically cover topics such as areas, volumes and rates of change. They often involve having to establish a suitable formula in one variable and then differentiating to find a maximum or minimum value. This is known as Optimisation.

It is important to check the validity of any solutions as often an answer is either nonexistent (e.g. a negative length or time) or outside an acceptable interval.

Optimisation

Optimisation is the term we use to describe a simple process of finding a maximum or minimum value for a given situation. Normally we would have to graph functions and then use our graph to establish a maximum or minimum.

As we have learned previously, differentiation allows us to quickly find the required value and is the expected manner of solving problems like this in Higher mathematics.

Drawing graphs would not be an acceptable solution.

Optimisation

Problems posed will often involve more than one variable. The process of differentiation requires that we rewrite or re-arrange formulae so that there is only one variable – typically *x*.

Questions are therefore multi-part where the first part would involve establishing a formula in *x* from a given situation, part 2 would involve the differentiation and validation of an acceptable answer, with part 3 the solution to the problem set.

You would do well to note that although the first part is important, you can normally expect to complete the rest of the question even when you cannot justify a formula in part 1.

Optimization : Maxima and Minima

Copy the following:

- Differentiation is most commonly used to solve problems by providing a "best fit" solution.

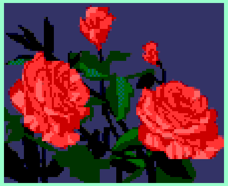
- Maximum and minimum values can be obtained from the **Stationary Points** and their nature.

- In exams you may be asked to "prove" a particular formula is valid. Even if you cannot prove this **USE THIS FORMULA TO ANSWER THE REST OF THE QUESTION.**

Example 1 (Formula Given)

The “Smelly Place” Garden Centre has a model train which is used to show visitors around the many displays of plants and flowers.

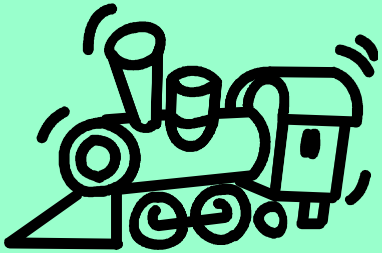
It has been established that the cost per month, C, of running the train is given by:

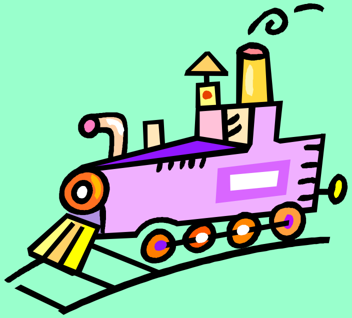


$$C = 50 + 25V + \frac{400}{V}$$

where V is the speed in miles per hour.

Calculate the speed which makes the cost per month a minimum and hence calculate this cost.





Example 1 $$C = 50 + 25V + \frac{400}{V}$$

Solution:

1. Find derivative, remembering to prepare for differentiation.

$$C = 50 + 25V + 400V^{-1}$$

$$\frac{dC}{dV} = 25 - 400V^{-2}$$

$$\frac{dC}{dV} = 25 - \frac{400}{V^2}$$

Example 1

$$C = 50 + 25V + \frac{400}{V}$$

Solution:

2. Make statement then set derivative equal to zero.

$$\frac{dC}{dV} = 25 - \frac{400}{V^2}$$

At SP's dC/dV = 0

$$0 = 25 - \frac{400}{V^2}$$

$$\frac{400}{V^2} = \frac{25}{1}$$

Cross Multiply

$$400 = 25V^2$$

$$V^2 = 16$$

$$V = 4$$

Reject –ve speed

Example 1

$$C = 50 + 25V + \frac{400}{V}$$

Solution:

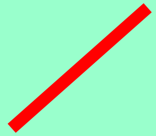
3. Justify nature of each SP using a nature table.

$$\frac{dC}{d(1)} = 25 - \frac{400}{(1)^2} = -375$$

$$\frac{dC}{d(5)} = 25 - \frac{400}{(5)^2} = 17$$

4. Make statement

$$\frac{dC}{dV} = 25 - \frac{400}{V^2} \qquad V = 4$$

X	4^-	4	4^+
$\frac{dC}{dV}$	-ve	**0**	+ve
Slope			

A minimum tp occurs and so cost is minimised at V = 4.

Example 1 $$C = 50 + 25V + \frac{400}{V}$$

Solution:

4. Sub x-value found into original expression to find corresponding minimum value.

(DO NOT USE dy/dx)

A minimum tp occurs and so cost is minimised at V = 4.

$$C = 50 + 25V + \frac{400}{V}$$

For V = 4:

$$C = 50 + 25(4) + \frac{400}{4}$$

$$C = 50 + 100 + 100$$

C = £250

5. Answer Question

Minimum cost is £250 per month and occurs when speed = 4mph

Heinemann , p.115, EX 6R
Q3

Example 2 (Perimeter and area)

A desk is designed which is rectangular in shape and which is required in the design brief to have a perimeter of 420 cm.

If $\boldsymbol{x}$ is the length of the base of the desk:

(a) Find an expression for the area of the desk, in terms of $\boldsymbol{x}$.

(b) Find the dimensions which will give the maximum area.

(c) Calculate this maximum area.

Example 2

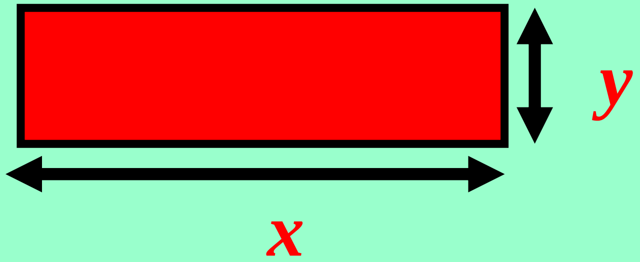
Solution to (a):

1. Start with what you know

Lets call breadth ***y***, so:

2. In order to differentiate we need expression only involve X’s. Use other information to find an expression for Y.

3. Change subject to y.



Area = length x breadth

Area = x x y

Perimeter = 2x + 2y

$$\Rightarrow 420 = 2x + 2y$$

$$\Rightarrow 2y = 420 - 2x$$

Divide by 2

$$\Rightarrow y = 210 - x$$

Example 2

Area = x x y $\Rightarrow y = 210 - x$

Solution to (a):

4. Now substitute this expression for y into our original expression for AREA.

Area = x x (210 – x)

$$Area = A(x) = 210x - x^2$$

Solution to (b):

5. To find dimensions giving maximum or minimum first find derivative.

$$\Rightarrow A'(x) = 210 - 2x$$

6. Make statement and set derivative equal to zero.

At SP's A'(x) = 0

$$\Rightarrow 210 - 2x = 0$$

$$\Rightarrow 2x = 210$$

$$\Rightarrow x = 105$$

Example 2

Solution to (a):

7. Now use your expression for y to find other dimension.

Area = ***x*** x ***y***

$\Rightarrow y = 210 - x$

$\Rightarrow x = 105$

$\Rightarrow y = 210 - 105 = 105$

Example 2

Solution:

$$\Rightarrow A'(x) = 210 - 2x$$

X = 105 and Y = 105

8. Must justify maximum values using nature table.

$$A'(100) = 210 - 2(100) = 10$$

$$A'(110) = 210 - 2(110) = -10$$

X	→	105	→
$A'(x)$	+ve	**0**	-ve
Slope	/	—	\

9. Make statement

The area is maximised when length is 105cm and breadth is 105 cm.

Example 2

Solution to (c):

10. To find actual maximum area sub maximum x-value just found into expression for area.

(DO NOT USE dy/dx)

The area is maximised when length is 105cm and breadth is 105 cm.

$$Area = A(x) = 210x - x^2$$

$$Area_{\max} = 210(105) - (105)^2$$

$$Area_{\max} = 22,050 - 11,025$$

$$Area_{\max} = 11,025\ cm^2$$

PROOF

Heinemann , p.112, EX 6Q
Q1, 2, 5

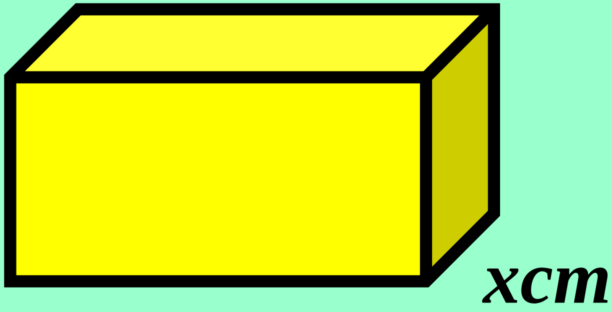
Example 3 (Surface Area and Volume)

A cuboid of volume 51.2 cm^3 is made with a length 4 times its breadth.

If ***x cm*** is the breadth of the base of the cuboid:

(a) Find an expression for the surface area of the cuboid, in terms of ***x***.

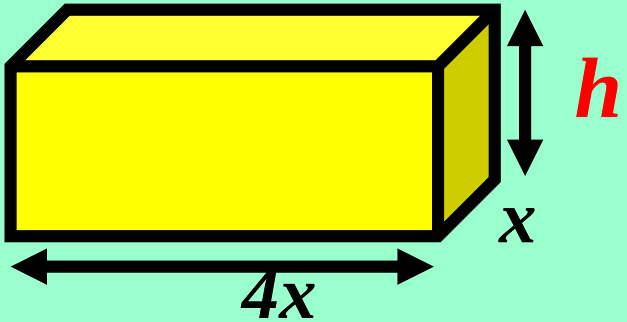
(b) Find the dimensions which will give the minimum surface area and calculate this area.



Example 3

Solution to (a):

1. Start with what you know



Surface Area = sum of area of faces

= 2(4x x h) + 2(x x h) + 2(4x x x)

= $10xh + 8x^2$

Lets call height ***h***, so:

Volume = 4x x x x h

$\Rightarrow 51.2 = 4x^2 \times h$

2. In order to differentiate expression must only involve X's. Use other information to find an expression for h.

3. Change subject to h.

$$\Rightarrow h = \frac{51.2}{4x^2} = \frac{12.8}{x^2}$$

Example 2

Surface Area = sum of area of faces

= **10xh + 8x²**

Solution to (a):

$$\Rightarrow h = \frac{12.8}{x^2}$$

4. Now substitute this expression for y into our original expression for SURFACE AREA.

$$A(x) = \left(10 \times x \times \frac{12.8}{x^2}\right) + \left(8x^2\right)$$

$$A(x) = \left(\frac{128x}{x^2}\right) + \left(8x^2\right)$$

$$A(x) = 8x^2 + \frac{128}{x}$$

Example 3

Solution to (b):

5. To find dimensions giving maximum or minimum first find derivative.

6. Make a statement and set derivative equal to zero.

$$A(x) = 8x^2 + \frac{128}{x}$$

$$A(x) = 8x^2 + 128x^{-1}$$

$$A'(x) = 16x - \frac{128}{x^2}$$

At SP's A'(x) = 0

$$\Rightarrow 16x - \frac{128}{x^2} = 0$$

Multiply by $\boldsymbol{x}^2$

$$\Rightarrow 16x^3 - 128 = 0$$

$$\Rightarrow 16x^3 = 128$$

$$\Rightarrow x^3 = 8$$

$$\Rightarrow x = \sqrt[3]{8} = 2$$

Example 3

Solution to (b):

7. To find corresponding height substitute x value just found into expression for ***h*** found in step 2.

$$h = \frac{12.8}{x^2}$$

$$\Rightarrow h = \frac{12.8}{(2)^2} = \frac{12.8}{4} = 3.2$$

So dimensions minimising area:

Breadth = 2 cm

Length = 8 cm

Height = 3.2 cm

Example 3

$$A'(x) = 16x - \frac{128}{x^2}$$

$$\Rightarrow x = \sqrt[3]{8} = 2$$

Solution:

8. Must justify minimum values using nature table.

$$A'(1) = 16 - 128 = -112$$

$$A'(4) = 16(4) - \frac{128}{16} = 56$$

X	→	2	→
$A'(x)$	-ve	**0**	+ve
Slope	\	_	/

9. Make statement

The surface area is minimised when the breadth is 2 cm

Example 3

Solution to (c):

10. To find <u>actual</u> maximum area sub maximum x-value just found into expression for area.

(DO NOT USE dy/dx)

The surface area is minimised when the breadth is 2 cm

$$A(x) = 8x^2 + \frac{128}{x}$$

$$Area_{\min} = 8(2)^2 + \frac{128}{2}$$

$$Area_{\min} = 32 + 64$$

$$Area_{\min} = 96\ cm^2$$

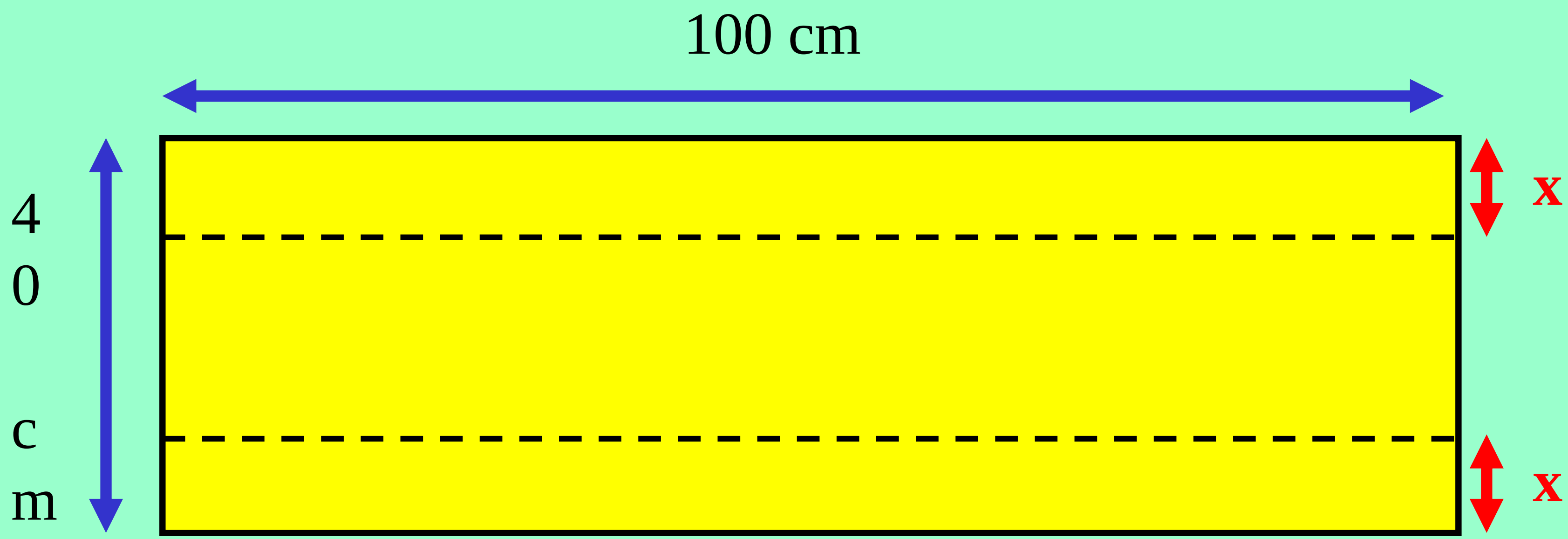
PROOF

Heinemann , p.112, EX 6R
Q1, 2, 5

Example 4 (ADDITIONAL - NOT ESSENTIAL)

A channel for carrying cables is being dug out at the side of the road.

A flat section of plastic is bent into the shape of a gutter and placed Into the channel to protect the cables.



The dotted line represents the fold in the plastic, x cm from either end.

(a) Show that the volume of each section of guttering is $V(x) = 200x(20 - x)$

(b) Calculate the value of x which gives the maximum volume of gutter and find this volume.

Example 2 (a) Show that the volume of each section of guttering is

$$V(x) = 200x(20 - x)$$

Solution to part (a):

Require **volume** so:

Length = 100 cm

Breadth = (40 – 2x) cm

Height = "folded bit" = x cm

$$V = l \times b \times h$$

$$V = 100 \times (40 - 2x) \times x$$

$$V = 100x(40 - 2x)$$

Factorise bracket

$$V = 100x \times 2(20 - x)$$

$$V = 200x(20 - x)$$ as required.

Example 2

(b) Calculate the value of x which gives the maximum volume of gutter. $V(x) = 200x(20 - x)$

Solution to part (b):

$$V(x) = 200x(20 - x)$$

Maximum and minimum will occur at SP's

$$V(x) = 4000x - 200x^2$$

Must prepare for differentiation

$$V'(x) = 4000 - 400x$$

At SP's $V'(x) = 0$

Must make this statement

$$0 = 4000 - 400x$$

$$0 = 400(10 - x)$$

$$0 = (10 - x) \longrightarrow$$ **x = 10**

Now prove this is a maximum tp

Example 2

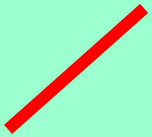
(b) Calculate the value of x which gives the maximum volume of gutter. $V(x) = 200x(20 - x)$

Solution to part (b):

x = 10

Now prove this is a maximum tp

$$V'(x) = 4000 - 400x$$

X	10^{-}	10	10^{+}
$\frac{dV}{dx}$	+ve	0	-ve
Slope			

So when x = 10 we have a maximum

Example 2

(b) Calculate the value of x which gives the maximum volume of gutter. $V(x) = 200x(20 - x)$

Solution to part (b):

So when x = 10 we have a maximum

$$V(x) = 200x(20 - x)$$

Now calculate maximum volume

$$V(10) = 200 \times 10 \times (20 - 10)$$

$$V(10) = 20000 cm^3$$

So maximum volume of guttering is 20,000 cm^3.

Make statement